4.1

Concept of Complex Numbers

Learning Objectives:

- To introduce the concept of a complex numbers through real numbers and the imaginary unit
- To define the conjugate of a complex number

  AND

- To practice the related problems

Imaginary Numbers

The equation

$$x^2 + 9 = 0 \qquad \ldots \qquad (1)$$

has no real number solutions because the square of a real number is always positive. The solution of this equation is given by

$$x = \sqrt{-9} = \sqrt{-1 \cdot 9} = \sqrt{-1} \cdot 3 = 3 \cdot \sqrt{-1}$$

The square root of a negative number, such as $\sqrt{-1}$, $\sqrt{-5}$, is called an *imaginary number*. We may write $\sqrt{-5}$ as $\sqrt{5} \cdot \sqrt{-1}$. It is convenient to introduce the symbol $i = \sqrt{-1}$ and write $\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1} = i\sqrt{5}$. Then the solution of the equation (1) is given by

$$x = 3i$$

The symbol i has the property $i^2 = -1$. It is the solution of the equation

$$x^2 + 1 = 0 \qquad \ldots \qquad (2)$$

For higher integral powers of i we have

$$i^3 = i^2 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = (-1) \times (-1) = 1$$

$$i^5 = i^4 \cdot i = i$$

We note that i^n has only four possible values: $1, i, -1, -i$. They correspond to values of n which divided by 4 leave the remainders 0,1,2,3.

Complex Numbers

An expression $x + iy$ is called a *complex number* where x and y are real numbers and i is the imaginary unit with the property $i^2 = -1$.

If $z = x + iy$, then the first term x is called the *real part* of the complex number z, denoted by $Re(z)$ and y is called the *imaginary part of* z *denoted by* $Im(z)$.

If $x = 0$, the number is said to be *purely imaginary*;
if $y = 0$, it is *real*.

Zero is the only number which is at once real and purely imaginary.

Complex numbers may be thought of as including all real numbers. For example, 8 can be written as $8 + 0i$. The real numbers occur when $y = 0$. When $y \neq 0$, we have complex numbers.

A complex number is denoted by z and the set of all complex numbers is denoted by C.

Thus, $$\boldsymbol{C = \{z : z = x + iy; x, y \epsilon R, i = \sqrt{-1}\}}$$

Two complex numbers are *equal* if and only if they have the same real part and the same imaginary part. Thus, the equality of two complex numbers $a + ib$ and $c + id$ imply $a = c$ and $b = d$.

Example: Find x and y if $3x + 4i = 12 - 8yi$.

Solution When two complex numbers are equal, their real parts are equal and their imaginary parts are equal.

$$3x = 12 \quad , \quad 4 = -8y$$

$$\Rightarrow x = 4 \quad , \quad y = -\frac{1}{2}$$

Example: Find x and y if $(4x - 3) + 7i = 5 + (2y - 1)i$.

Solution Equating the real and imaginary parts,

$$4x - 3 = 5 \ , \ 2y - 1 = 7$$

$$\Rightarrow 4x = 8 \quad , \quad 2y = 8$$

$$\Rightarrow x = 2 \quad , \quad y = 4$$

Conjugate of a complex number:

The *conjugate* of a complex number $\alpha + i\beta$ is the complex number $\alpha - i\beta$, obtained by replacing i by $-i$. The process of replacing i by $-i$ is called *complex conjugation*.

For example, $2 - 3i$ is the complex conjugate of $2 + 3i$.

The complex conjugate of a complex number z is denoted by $\bar{z}$. If $z = x + iy$, then $\bar{z} = x - iy$. Notice that

$z = x + iy \Rightarrow \bar{z} = x - iy \Rightarrow \bar{\bar{z}} = x + iy = z$. Thus,

$$\boldsymbol{\bar{\bar{z}} = z} \textbf{ for all } \boldsymbol{z \epsilon C}.$$

P1:

Find the product of $-i, 2i$ and $\left(-\frac{1}{8}i\right)^3$.

Solution:

We have, $-i, 2i$ and $\left(-\frac{1}{8}i\right)^3$

$$(-i)(2i)\left(-\frac{1}{8}i\right)^3 = 2 \times \frac{1}{8\times8\times8} \times i^5 = \frac{1}{256} \times (i^2)^2 \times i = \frac{1}{256}i$$

P2:

If $4x + i(3x - y) = 3 - 6i$, where x and y are real numbers, then find the values of x and y.

Solution:

We have, $4x + i(3x - y) = 3 - 6i$

Two complex numbers are equal if their real and imaginary parts are equal.

Thus, $4x = 3$ and $3x - y = -6$

$$\Rightarrow x = \frac{3}{4} \text{ and } 3\left(\frac{3}{4}\right) - y = -6 \Rightarrow y = \frac{9}{4} + 6 \Rightarrow y = \frac{33}{4}$$

$$\therefore x = \frac{3}{4} \text{ and } y = \frac{33}{4}$$

P3:

Evaluate $(i^{77} + i^{70} + i^{87} + i^{414})^3$.

Solution:

We have, $(i^{77} + i^{70} + i^{87} + i^{414})^3$

$$= (i^{4\cdot19+1} + i^{4\cdot17+2} + i^{4\cdot21+3} + i^{4\cdot103+2})^3$$

$$= (i^1 + i^2 + i^3 + i^2)^3 = (i - 1 - i - 1)^3$$

$$= (-2)^3 = -8$$

P4:

Evaluate $1 + i^2 + i^4 + i^6 + \cdots + i^{20}$

Solution:

$$1 + i^2 + i^4 + i^6 + \cdots + i^{20}$$

$$= (1 + i^2) + (i^4 + i^6) + (i^8 + i^{10}) + \cdots + (i^{16} + i^{18}) + i^{20}$$

$$= (1 + i^2) + i^4(1 + i^2) + i^8(1 + i^2) + \cdots + i^{16}(1 + i^2) + i^{20}$$

$$= i^{20} = 1$$

IP1:

Evaluate $i^{49} + i^{68} + i^{89} + i^{110}$.

Solution:

We have,

$$\begin{aligned} i^{49} + i^{68} + i^{89} + i^{110} &= i^{4\cdot 12+1} + i^{4\cdot 17+0} + i^{4\cdot 22+1} + i^{4\cdot 27+2} \\ &= i^1 + i^0 + i^1 + i^2 \\ &= i + 1 + i - 1 = 2i \end{aligned}$$

IP2:

If $(2x - y) + i = 3 - i(2y - x)$, then find the values of x and y.

Solution:

We have $(2x - y) + i = 3 - i(2y - x)$

Comparing the real and the imaginary parts on both sides, we get

$$2x - y = 3 \quad \dots\dots\dots (1)$$

and $$-2y + x = 1 \quad \dots\dots\dots (2)$$

Solving (1) and (2), we get $x = \frac{5}{3}, y = \frac{1}{3}$.

IP3:

Evaluate

i) $\left(-\sqrt{-1}\right)^{4n+3}, n \epsilon N$

ii) $i^n + i^{n+1} + i^{n+2} + i^{n+3}, n \epsilon N$

Solution:

i) $\left(-\sqrt{-1}\right)^{4n+3} = (-i)^{4n+3} = (-i)^{4n}(-i)^3$

$$= \{(-i)^4\}^n(-i)^3 = 1 \times -i^3 = i$$

ii) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n(1 + i + i^2 + i^3)$

$$= i^n(1 + i - 1 - i) = 0$$

IP4:

Evaluate $1 + i^2 + i^4 + i^6 + \cdots + i^{22}$

Solution:

$$1 + i^2 + i^4 + i^6 + \cdots + i^{22}$$

$$= (1 + i^2) + (i^4 + i^6) + (i^8 + i^{10}) + \cdots + (i^{16} + i^{18})$$
$$+(i^{20} + i^{22})$$

$$= (1 + i^2) + i^4(1 + i^2) + i^8(1 + i^2) + \cdots + i^{16}(1 + i^2)$$
$$+i^{20}(1 + i^2)$$

$$= 0$$

Note:

$$\mathbf{1 + i^2 + i^4 + i^6 + \cdots + i^{2n} = \begin{cases} 0, & if\ n\ is\ odd \\ 1, & if\ n\ is\ evn \end{cases}}$$

1. Evaluate the following:

a. i^{30}

b. i^{11}

c. i^{40}

d. i^{135}

e. i^{-999}

f. $i^{37} + \frac{1}{i^{67}}$

g. $\left(i^{41} + \frac{1}{i^{257}}\right)^{9}$

h. $i^{49} + i^{68} + i^{89} + i^{110}$

i. $i^{30} + i^{80} + i^{120}$

j. $\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}$

k. $\left(i^{18} + \left(\frac{1}{i}\right)^{24}\right)^{3}$

2. Find x and y such that $2x - iy = 4 + 3i$

3. Find the complex conjugate of $z = a + 2i + 3ib$.